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A Class of Soluble Groups

NIGEL BOSTON

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

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INTRODUCTION

In this paper a problem in group theory is solved to produce an interesting class of groups. The motivation, however, comes from algebraic number theory and in particular the Artin representation.

To describe this representation we follow [5, Chap. IV and VI]. Let K and L be local fields, i.e., fields complete with respect to a discrete valuation and with perfect residue field. Let L have valuation ring A_L and valuation v_L extending the valuation on K. Suppose that L/K is a finite Galois extension with Galois group H.

Then H has a normal series consisting of the ramification groups $H_j = \{\sigma \in H : v_L(\sigma x - x) \geqslant j+1 \ \forall\, x \in A_L\}$, $j = -1, 0, 1, 2, \ldots$, which produces a character of H, due to Artin, as described in Theorem 1 below.

The notation adopted in this paper is as follows. If G is a finite group, then $\mathrm{Irr}(G)$ denotes its set of irreducible characters. 1_G, r_G, and $u_G = r_G - 1_G$ are its principal, regular, and augmentation characters, respectively, whilst unless otherwise stated χ_G and χ^G will mean the restriction of χ to G and the character of G induced from χ.

THEOREM 1. *Let G be any finite group and $G = G_0 \rhd G_1 \rhd \cdots \rhd G_m = 1$ any normal series of G, i.e., $G_j \lhd G$, $j = 0, 1, \ldots, m$.*

Set $i(x) = \sup\{j : x \in G_j\} + 1$ if $1 \neq x \in G$,

$$i(1) = -\sum_{x \neq 1} i(x) \qquad (\textit{orthogonality with } 1_G).$$

Then $\exists\, \lambda \in \mathbb{Z}^+$ such that $-\lambda i$ is a character of G.

Proof. Consider

$$\begin{aligned} (u_{G_j})^G(x) &= \quad 0 \qquad \text{if} \quad x \notin G_j, \\ &= -\frac{g_0}{g_j} \qquad \text{if} \quad 1 \neq x \in G_j, \text{ where } g_j = \mathrm{card}(G_j). \end{aligned}$$

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So if $x \neq 1$, then

$$i(x) = -\sum_{j=0}^{m-1} \frac{g_j}{g_0} (u_{G_j})^G(x) \qquad (1),$$

true also for $x = 1$ by orthogonality with 1_G. Taking $\lambda = g_0$, the theorem is proved.

It may be possible in some cases to take $\lambda < g_0$ and still obtain a character $-\lambda i$. In particular if G_j is the jth ramification group of the extension L/K described earlier, then Artin's result states that we may take $\lambda = 1$ in Theorem 1.

If G is an abstract soluble group, then from a group theorist's point of view a natural choice for G_j is the jth derived subgroup $G^{(j)}$, and this paper classifies those finite soluble groups G for which we can take $\lambda = 1$ with this choice of $\{G_j\}$. We call these *λ-groups* and call the smallest m for which $G^{(m)} = 1$ the *length* of G.

1. Main Theorem

THEOREM 2. *A λ-group G must be one of the following*:

(i) *abelian*,

(ii) *Frobenius of length* 2,

(iii) *a CN-group of length* 3 *satisfying $G/G' \cong C_2$, G'' is a 2-group, G'/G'' is cyclic of odd order, and both G/G'' and G' are Frobenius. Thus a λ-group is always a CN-group. Conversely, a finite group of type* (i), (ii), *or* (iii) *is a λ-group.*

We explain these terms. A finite group G is called *Frobenius* if it has a proper subgroup H such that $x^{-1}Hx \cap H = 1 \ \forall\, x \in G - H$. A finite group G is called a *CN-group* if $C_G(x)$ is nilpotent whenever $1 \neq x \in G$. For an account of the recent importance of these groups see [1].

The proof of Theorem 2 is in three stages. First we obtain necessary and sufficient conditions for a group to be a λ-group in terms of the degrees of its irreducible characters. This reduces the problem to constructing groups from constraints on their characters.

Next the λ-groups of length 2 are classified with the consequence that no λ-group can have length 4 or more. This leaves the classification of λ-groups G of length 3, accomplished by studying the action of G/G' on G''.

2. Character Degrees of λ-Groups

Set $\rho_G = -i$ in Theorem 1 (where G_j is $G^{(j)}$). Thus G is a λ-group if and only if ρ_G is a character of G. Derived subgroups of a λ-group are λ-groups by:

Lemma 1. *If G is a λ-group, then G' is a λ-group.*

Proof. Note that by Eq. (1) if ρ_G is a character of G, then so is $\rho_G - u_G$ and moreover $\rho_{G'} = (\rho_G - u_G)_{G'}$, implying that $\rho_{G'}$ is a character of G'.

We consider λ-groups as being built up from λ-groups of shorter length. In other words let $H = G'$ be a λ-group and seek conditions for G to be a λ-group.

For this we need Clifford's theorem [3, p. 79]. This says that if $\chi \in \mathrm{Irr}(G)$, then χ_H has the form $e \sum_{i=1}^{r} \psi_i$, where $e \in \mathbb{Z}^+$ and the ψ_i are a complete set of conjugates of some $\psi \in \mathrm{Irr}(H)$. This is reminiscent of how prime ideals split in normal extensions of number fields, taking e to be ramification index in this analogy. Moreover $er \mid [G:H]$ [3, p. 190], so set $f = [G:H]/er$ in analogy to residue degree. This is a function of ψ [3, p. 95], and we have:

Lemma 2. *ρ_G is a character of G if and only if $f(\psi) \mid (\rho_H, \psi)_H \; \forall\, \psi \in \mathrm{Irr}(H)$.*

Proof. From Eq. (1),

$$\rho_G = \frac{h}{g}\rho_H^G + u_G \tag{2}$$

where $h = \mathrm{card}(H)$, $g = \mathrm{card}(G)$. Therefore, ρ_G is a character of $G \Leftrightarrow (\rho_G, \chi)_G \in \mathbb{Z} \;\forall\, \chi \in \mathrm{Irr}(G) \Leftrightarrow (h/g)(\rho_H^G, \chi)_G \in \mathbb{Z} \;\forall\, \chi \in \mathrm{Irr}(G) \Leftrightarrow (g/h) \mid (\rho_H, \chi_H)_H \;\forall\, \chi \in \mathrm{Irr}(G) \Leftrightarrow (g/h) \mid e \sum_{i=1}^{r} (\rho_H, \psi_i)_H \;\forall\, \chi \in \mathrm{Irr}(G)$ following Clifford's theorem $\Leftrightarrow (g/h) \mid er(\rho_H, \psi)_H \;\forall\, \psi \in \mathrm{Irr}(H)$, and the result follows.

To study ρ_G for groups of length 2 and 3, let $K = H'$ and $k = \mathrm{card}(K)$. Then K is abelian and 2 cases arise:

Lemma 3. (a) *If $1_H \neq \psi \in \mathrm{Irr}(H)$ is linear, then $(\rho_H, \psi)_H = 1$.*

(b) *If $\psi \in \mathrm{Irr}(H)$ and ψ_K is a sum of nonprincipal linear characters, then $(\rho_H, \psi)_H = (k/h + 1)\,\psi(1)$.*

Proof. By Eq. (2),

$$\begin{aligned}(\rho_H, \psi)_H &= \frac{k}{h}(\rho_K^H, \psi)_H + (u_H, \psi)_H \\ &= \frac{k}{h}(\rho_K, \psi_K)_K + \psi(1).\end{aligned} \tag{3}$$

(a) If ψ is nonprincipal and linear, then $\psi_K = 1_K$, $\psi(1) = 1$, and the result follows from (3).

(b) By Clifford's theorem, ψ_K has decomposition $e\sum_{i=1}^{r} \zeta_i$, say, with ζ_i linear. Then $\psi(1) = er \Rightarrow$ by (3), $(\rho_H, \psi)_H = (k/h)\,\psi(1)(\rho_K, \zeta)_K + \psi(1)$, but by (a) $(\rho_K, \zeta)_K = 1$.

This produces an effective criterion for a group of length 2 or 3 to be a λ-group. Note that any abelian group is a λ-group ($\rho_G = u_G$).

LEMMA 4. (a) *Let G be a finite soluble group of length* 2. *Then G is a λ-group if and only if its nonlinear irreducible characters have degree* $[G: G']$.

(b) *Let G be a finite soluble group of length* 3. *Then G is a λ-group if and only if its nonlinear irreducible characters are of* 2 *types*:

(i) *those of degree* $[G: G']$, *which restrict to sums of linear characters of* G', *and*

(ii) *those of degree an integer multiple of* $[G: G'']/(1 + [G': G''])$, *which restrict to sums of nonlinear irreducible characters of* G'.

Proof. (a) Since all irreducible characters of G' are linear, the result follows from Lemmas 2 and 3(a).

(b) From Lemmas 2 and 3. For (ii), note that the condition $f(\psi) \mid (\rho_H, \psi)_H$ becomes

$$\frac{g}{h}\frac{\psi(1)}{\chi(1)} \,\Big|\, \left(\frac{k}{h} + 1\right)\psi(1),$$

where χ is an irreducible constituent of ψ^G.

COROLLARY 1. *If G is a λ-group of length* 2, *then* $\operatorname{card}(G')$ *and* $[G: G']$ *are coprime.*

Proof. If $\operatorname{Irr}(G)$ has c nonlinear characters and $d = [G: G']$, then $\operatorname{card}(G) = \sum_{\operatorname{Irr}(G)} (\chi(1))^2 = d + cd^2$, whence $\operatorname{card}(G') = 1 + cd$.

COROLLARY 2. *A nilpotent λ-group G must be abelian.*

Proof. If G has length $m > 1$, then by Lemma 1 $G^{(m-2)} = F$ is a λ-group of length 2. F is also nilpotent, so $\operatorname{card}(F')$ and $[F: F']$ cannot be coprime, contradicting Corollary 1.

3. λ-GROUPS OF LENGTH 2

LEMMA 5. *Let G be a finite soluble group of length* 2. *Then G is a λ-group if and only if G is Frobenius.*

Proof. First it is well known that the irreducible characters of a Frobenius group of length 2 satisfy Lemma 4(a); see, e.g., [3, p. 94].

Conversely, by Corollary 1 and Schur–Zassenhaus, there exists a subgroup F of order $d=[G: G']$ and index $1+cd=n$, complementing G'.

Consider permutation

$$\pi(x)=\begin{pmatrix} Ft_1 & \cdots & Ft_n \\ Ft_1x & \cdots & Ft_nx \end{pmatrix},$$

where $\{t_i\}$ is a transversal of F in G and $t_1=1$. This gives a representation of G with character $\operatorname{card}\{i: Ft_i=Ft_ix\}=\operatorname{card}\{i: x\in t_i^{-1}Ft_i\}=1_F^G$. Let $(1_F^G)_F=\phi$. We want to show that $\phi(x)=1$ if $1\neq x\in F$, since then $t_i^{-1}Ft_i\cap F=1$ if $i\neq 1$, and the lemma follows.

Take $\chi\in \operatorname{Irr}(G)$ and analyse χ_F. Then

(i) the d linear characters of G restrict to give all the d irreducible characters of F. For if $\chi_F=\zeta_F=\psi$ where $\chi, \zeta\in\operatorname{Irr}(G)$ are linear, then $(\chi, \psi^G)_G=1=(\zeta, \psi^G)_G$. But ψ^G has degree $1+cd$ and there are only d linear characters of G, so $\chi=\zeta$.

(ii) If χ has degree d, then $\chi_F=r_F$. Since $F\cap G'=1$, $\chi_F(x)=0$ if $1\neq x\in F$. Therefore χ_F is an integer multiple of r_F. Comparing degrees, $\chi_F=r_F$.

Therefore, if 1_F^G decomposes as $\sum c_i\chi_i$, $\chi_i\in\operatorname{Irr}(G)$, then $c_i=(1_F, \chi_{i_F})_F=1$ if χ_i is nonlinear or 1_G, else $=0$.

Suppose ϕ decomposes as $\sum k_j\psi_j$, $\psi_j\in\operatorname{Irr}(F)$. Then $k_j=(\phi, \psi_j)_F=(1_F^G, \psi_j^G)_G=(1_G, \psi_j^G)_G+\sum(\chi_i, \psi_j^G)_G$, summed over nonlinear $\chi_i\in\operatorname{Irr}(G)=(1_F, \psi_j)_F+\sum(\chi_{i_F}, \psi_j)_F=c+1$ if $\psi_j=1_F$, else $=c$.

Therefore, $\phi=cr_F+1_F$, whence $\phi(x)=1$ if $x\neq 1$.

If $x\in G-G'$, $C_G(x)\cong G/G'$, whilst if $x\in G'-1$, $C_G(x)=G'$. These are both abelian groups so G is a CN-group. It is known that if H is a Frobenius group of length 2, then H/H' is cyclic [2, p. 339]. From this we deduce:

LEMMA 6. *A λ-group G has length at most 3. Moreover, if G has length 3, then G/G' and G'/G'' are both cyclic, and G/G'' and G' are both Frobenius.*

Proof. If G is a λ-group of length 3, then G' and G/G'' are λ-groups by Lemmas 1 and 4, respectively, so both are Frobenius by Lemma 5. By the above comments G/G' and G'/G'' are cyclic. This rules out the possibility of a λ-group of length 4 or more by Zassenhaus's theorem [4, p. 246], which says that if F is a finite soluble group with F'/F'' and F''/F''' both cyclic, then $F''=1$.

4. λ-Groups of Length 3

For this section assume that G is a λ-group of length 3 with $G' = H$ and $G'' = K$. First we find $[G:H]$.

Lemma 7. *If $H < J \leqslant G$, then $J' = H$ and J is a λ-group.*

Proof. J/K is a Frobenius group of length 2, implying that $(J/K)' = H/K$ and $J' = H$. Comparing the normal series $G \rhd H \rhd K \rhd 1$ and $J \rhd H \rhd K \rhd 1$, $\rho_J = (\rho_G)_J$, which is a character of J.

In applications J will be an inertia group $I_G(\psi) = \{g \in G: \psi(g^{-1}xg) = \psi(x)\ \forall\, x \in H\}$, $\psi \in \operatorname{Irr}(H)$. Note that $H \leqslant I_G(\psi)$.

Lemma 8. *If $\exists\ 1_H \neq \psi \in \operatorname{Irr}(H)$ satisfying $I_G(\psi) = G$, then $[G:H] = 2$.*

Proof. By [3, p. 186], since $I_G(\psi) = G$ and G/H is cyclic, ψ is extendible to G, i.e., $\exists\, \chi \in \operatorname{Irr}(G)$ with $\chi_G = \psi$. If ψ is linear, then ψ^G is irreducible by Lemma 4 implying $I_G(\psi) = H$. Thus χ must have degree $[H:K] = 1 + cd$, say, where $[G:H] = d$.

Applying Lemma 4(b), $1 + cd = M(d(1 + cd)/(1 + 1 + cd))$ for some $M \in \mathbb{Z}^+$, so $2 + cd = Md$, implying $d \mid 2$.

It therefore suffices to find such a ψ. For this we use Brauer's lemma [1, p. 66]. This says that if L is a cyclic group of permutations of both $\operatorname{Irr}(H)$ and the set of conjugacy classes of H, such that permuting the rows of the character table by $\sigma \in L$ produces the same result as permuting the columns, then the number of rows left fixed by L equals the number of columns left fixed by L.

Taking L to be G/H with action by conjugation, $\psi \in \operatorname{Irr}(H)$ lies in an orbit of length 1 if and only if $I_G(\psi) = G$. Thus by Lemma 8 it suffices to find a nontrivial conjugacy class of H left fixed by L in order to show $[G:H] = 2$. Let us assume one does not exist.

Note that in this case if $\operatorname{card}(G/H)$ is a prime power q^r, then counting the orbits of conjugacy classes, $\operatorname{card}(H) \equiv 1 \pmod q$. By Schur–Zassenhaus G splits over H, and L acts as a subgroup of G.

Lemma 9. *If prime $p \mid [G:H]$, then $p = 2$.*

Proof. By Lemma 7 we may assume $G/H \cong C_p$. Then any generator of L induces a fixed-point-free automorphism of H of order p. By Thompson's theorem [2, p. 339], H must be nilpotent, contradicting Corollary 2.

By Lemma 7 we now have only the case $[G:H] = 4$ to eliminate. Let $L = \langle \sigma \rangle$ of order 4, and let $\tau = \sigma^2$.

LEMMA 10. *Suppose H has odd order and let τ be any automorphism of H of order 2. Suppose $F = \{x \in H: \tau(x) = x\}$ lies in K. Then $1 \neq F \neq K$, and distinct elements of F cannot be conjugate in H.*

Proof. Define homomorphism $\phi: K \to K$ by $\phi(k) = \tau(k)\,k^{-1}$. Then $\ker \phi = F$, and since $\tau(\tau(k)\,k^{-1}) = k\tau(k)^{-1}$, $K/F \cong I = \operatorname{im} \phi \leqslant \{x \in K: \tau(x) = x^{-1}\}$. Since K has odd order, $I \cap F = 1$. Also $IF = K$, and we deduce that $I = \{x \in K: \tau(x) = x^{-1}\}$.

Now suppose $x = h^{-1}yh$, $x, y \in F$, $h \in H$. Then $x = \tau(h)^{-1}\, y\tau(h)$, whence $u = \tau(h)\,h^{-1} \in C_H(y) = K$. But $\tau(u) = u^{-1}$, so $u \in I \Rightarrow u = \tau(k)\,k^{-1}$ for some $k \in K \Rightarrow \tau(k^{-1}h) = k^{-1}h \Rightarrow k^{-1}h \in F \Rightarrow h \in K \Rightarrow x = y$.

Thus $F \neq K$. Also $F \neq 1$ by Thompson's theorem.

With our choice of τ, $F \leqslant K$ since G/K is Frobenius. Now if $k \in I$, one checks that $\sigma(k) \in I$, so σ is a fixed-point-free automorphism of I. By reasoning as in Lemma 10, $\sigma(x) = x^{-1}\ \forall\, x \in I$, implying $\tau(x) = x\ \forall\, x \in I$. This says that $K = F$, a contradiction completing the proof of:

LEMMA 11. $[G: H] = 2$.

Lemma 10 also enables us to prove:

LEMMA 12. *K is a 2-group.*

Proof. Suppose not. Let S be the unique Sylow 2-subgroup of K. Then $S \lhd G$, and by Lemma 4(b) G/S is a λ-group. So assume K has odd order. Note again that H has odd order, G splits over H, and take $L = \langle \tau \rangle$. As with Lemma 9, τ fixes a nontrivial $\psi \in \operatorname{Irr}(H)$. By Brauer's lemma τ fixes a nontrivial conjugacy class of H, which must lie in K since G/K is Frobenius. By Lemma 10 this conjugacy class contains some $x \notin F$.

Set $x = uv$, $1 \neq u \in I$, $v \in F$. x is conjugate to $\tau(x) = u^{-1}v$, i.e., $u^g v^g = u^{-1}v$ for some $g \in H$. $ug^{-1}ug = vg^{-1}v^{-1}g$. Therefore $u^{-1}v \in C_G(g) \cap K$. $u \neq 1 \Rightarrow \tau(x) \neq x \Rightarrow g \notin K \Rightarrow u^{-1}v = 1$, a contradiction.

If $x \in K - 1$ or $G - H$, then $C_G(x)$ is a 2-group and so is nilpotent. If $x \in H - K$, $C_G(x) \cong H/K$, abelian so nilpotent. Thus G is a CN-group.

For the converse, suppose G is of type (iii) in Theorem 2. To show that G is a λ-group, we apply Lemma 4(b). First $\operatorname{Irr}(G/G'')$ gives the irreducible characters of G of degrees 1 and 2. The remaining irreducible characters of G (those of type (ii)) are induced from G', when it suffices to note that e is an integer multiple of $2e/(1+e)$ if $e = [G': G'']$, since e is odd. The simplest example of such a group is S_4.

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